Algebra I

Back Paper

Instructions. All questions carry equal marks. Justify all your answers.

1. Define an equivalence relation on a set S. Explicitly write down an equivalence relation on the real plane $\mathbf{R}^2$ whose equivalence classes are parabolas parallel to $y = x^2$.
2. Define the order of an element in a group G. Prove or disprove: If $\theta : G \to H$ is a group homomorphism between two finite groups, then the order of $\theta(x)$ divides the order of x.
3. State and prove the Lagrange's theorem for finite groups.
4. Let x and y be two elements of a finite group G. Prove that the order of xy and yx are always equal.
5. Classify finite groups all of whose elements have order 2.
6. Define class equation of a finite group. Prove that the center of p group is non-trivial.
7. State the three Sylow's theorems for finite groups. Prove that any group of order 15 is cyclic.
8. Let H be a subgroup of a finite group G whose index is the smallest prime number dividing the order of G. Prove that H is normal in G.
9. Prove that the image of a Sylow subgroup is a Sylow subgroup under any surjective group homomorphism between two finite groups.
10. Let G be a finite p group (where p is a prime number). Let d be a divisor of the order of G. Prove that G has a subgroup of order d.